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Quantum-Logics-Valued Measure Convergence Theorem

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In this paper, the following quantum-logic valued measure convergence theorem is proved: Let $(L_1, 0, 1)$ be a Boolean algebra, $(L_2, \perp, \oplus, 0, 1)$ be a quantum logic and $\{\mu_n : n \in \mathbf{N}\}$ be a sequence of s-bounded $(L_2, \perp, \oplus, 0, 1)$-valued measures which are defined on $(L_1, 0, 1)$. If for each $a \in (L_1, 0, 1)$, $\{\mu_n(a)\}_{n\in\mathbf{N}}$ is an order topology $\tau_0^{L_2}$ Cauchy sequence, when $\{v(a)\}$ convergent to 0, $\{\mu_n(a)\}$ is order topology $\tau_0^{L_2}$ convergent to 0 for each $n \in \mathbf{N}$, where v is a nonnegative finite additive measure which is defined on $(L_1, 0,1)$, then when $\{v(a)\}$ convergent to 0, $\{\mu_n(a)\}$ are order topology $\tau_0^{L_2}$ convergent to 0 uniformly with respect to $n \in \mathbf{N}$.

KEY WORDS: quantum logics; effect algebras; measures.

1. INTRODUCTION

In 1994, Foulis and Bennett (1994) introduced the following quantum logic structure and called it the *effect algebra*.

Let L be a set with two special elements 0, 1, $\perp$ be a subset of $L \times L$, if $(a, b) \in \perp$, denote $a \perp b$, and let $\oplus : \perp \to L$ be a binary operation. We say that the algebraic system $(L, \perp, \oplus, 0, 1)$ is an *effect algebra* if the following axioms hold

- (i) (Commutative Law) If $a, b \in L$ and $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$.
- (ii) (Associative Law) If $a, b, c \in L$, $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- (iii) (Orthocomplementation Law) For each $a \in L$ there exists an unique $b \in L$ such that $a \perp b$ and $a \oplus b = 1$.
- (iv) (Zero-Unit Law) If $a \in L$ and $1 \perp a$, then $a = 0$.

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Let $(L, \perp, \oplus, 0, 1)$ be an effect algebra. If $a, b \in L$ and $a \perp b$ we say that a and b be *orthogonal*. If $a \oplus b = 1$ we say that b is the *orthocomplement* of a, and we write $b = a'$. Clearly $1' = 0$, $(a')' = a$, $a \perp 0$ and $a \oplus 0 = a$ for all $a \in L$. We say that $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. We may prove that $\leq$ is a partial ordering on L and satisfies that $0 \leq a \leq 1$, $a \leq b \Leftrightarrow b' \leq a'$ and $a \leq b' \Leftrightarrow a \perp b$ for $a, b \in L$.

If $a \leq b$, the element $c \in L$ such that $c \perp a$ and $a \oplus c = b$ is unique, and satisfies the condition $c = (a \oplus b')'$. It will be denoted by $c = b \ominus a$.

Let $F = \{a_i : 1 \leq i \leq n\}$ be a finite subset of L. If $a_1 \perp a_2$, $(a_1 \oplus a_2) \perp a_3, \ldots$ and $(a_1 \oplus a_2 \cdots \oplus a_{n-1}) \perp a_n$, we say that F is *orthogonal* and we define $\oplus F = a_1 \oplus a_2 \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ (by the commutative and associative laws, this sum does not depend of any permutation of elements). Now, if A is an arbitrary subset of L and $\mathcal{F}(A)$ is the family of all finite subsets of A, we say that A is *orthogonal* if F is orthogonal for every $F \in \mathcal{F}(A)$. If A is orthogonal, we define $\oplus A = \bigvee\{\oplus F : F \in \mathcal{F}(A)\}$, supposed that the supremum exists in $(L, \leq)$, and it is called the $\oplus$-*sum* of A. If A is an orthogonal subset of L and $B \subseteq A$, it is obviously that B is also orthogonal. If there exist $\oplus A$ and $\oplus B$, then $\oplus B \leq \oplus A$. Moreover, let $(a_i)_{i \in I}$ be an orthogonal subset of L, then we may prove (Mazario, 2001)

(1) If I is finite and $J \subseteq I$, then $(\oplus_{i \in J} a_i) \perp (\oplus_{i \in I \setminus J} a_i)$ and

$$(\oplus_{i \in I} a_i) = (\oplus_{i \in J} a_i) \oplus (\oplus_{i \in I \setminus J} a_i)$$

(2) If $J \subseteq I$ and there exists $a = \oplus_{i \in I} a_i$, $b = \oplus_{i \in J} a_i$, $c = \oplus_{i \in I \setminus J} a_i$, then $b \perp c$ and $a = b \oplus c$.
(3) If there exists $\oplus_{i \in M} a_i$ for all $M \subseteq I$ and $\{H_j : j \in J\}$ is a partition of I, then $A = \{\oplus_{i \in H_j} a_i : j \in J\}$ is an orthogonal subset of L, there exists $\oplus A$ and $\oplus A = \oplus_{i \in I} a_i$.
(4) If $(F_j)_{j \in J}$ is a family of finite and pairwise disjoint subsets of I, then the set $\{\oplus_{i \in Fj} a_i : j \in J\}$ is orthogonal in L.
(5) If $b_i \in L$ and $b_i \leq a_i$ for all $i \in I$, then $(b_i)_{i \in I}$ is an orthogonal subset of L.
(6) If $a \oplus b$ and $a \vee b$ exist, then $a \wedge b$ exists and $a \oplus b = (a \vee b) \oplus (a \wedge b)$.

If the partial order $\leq$ of effect algebra $(L, \perp, \oplus, 0, 1)$ defined as above is a lattice, then the effect algebra $(L, \perp, \oplus, 0, 1)$ is said to be a *lattice effect algebra*.

If for all $a, b \in L$, $a \leq b$ or $b \leq a$, then $(L, \perp, \oplus, 0, 1)$ is said to be a *totally ordered effect algebra*; if for all $a, b \in L$, satisfies that $a < b$, there exists $c \in L$ such that $a < c < b$, then $(L, \perp, \oplus, 0, 1)$ is said to be *connected*.

An effect algebra is *complete*, if for each orthogonal subset A of L, the $\oplus$-sum $\oplus A$ exists; if for each countable orthogonal subset B of L, the $\oplus$-sum $\oplus B$ exists, then we say that the effect algebra is σ-*complete*.

We say that the effect algebra L has the *sequential completeness property*, if for each orthogonal sequence $\{a_i\}$ of L, there is a subsequence $\{a_{i_k}\}$ of $\{a_i\}$ such that $\oplus_k a_{i_k}$ exists.

2. ORDER TOPOLOGY OF QUANTUM LOGICS

A partial ordered set $(\Lambda, \preceq)$ is said to be a *directed set*, if for all $\alpha, \beta \in \Lambda$, there exists $\gamma \in \Lambda$ such that $\alpha \preceq \gamma, \beta \preceq \gamma$.

If $(\Lambda, \preceq)$ is a directed set and for each $\alpha \in \Lambda$, $a_\alpha \in (L, \perp, \oplus, 0, 1)$, then $\{a_\alpha\}_{\alpha\in\Lambda}$ is said to be a *net* of $(L, \perp, \oplus, 0, 1)$.

Let $\{a_\alpha\}_{\alpha\in\Lambda}$ be a net of $(L, \perp, \oplus, 0, 1)$. Then we write $a_\alpha \uparrow$, when $\alpha \preceq \beta$, $a_\alpha \leq a_\beta$. Moreover, if a is the supremum of $\{a_\alpha : \alpha \in \Lambda\}$, i.e., $a = \vee\{a_\alpha : \alpha \in \Lambda\}$, then we write $a_\alpha \uparrow a$.

Similarly, we may write $a_\alpha \downarrow$ and $a_\alpha \downarrow a$.

If $\{u_\alpha\}_{\alpha\in\Lambda}, \{v_\alpha\}_{\alpha\in\Lambda}$ are two nets of $(L, \perp, \oplus, 0, 1)$, for $u \uparrow u_\alpha \leq v_\alpha \downarrow v$ means that $u_\alpha \leq v_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$ and $v_{\alpha\downarrow} v$. We write $b \leq u_\alpha \uparrow u$ if $b \leq u_\alpha$ for all $\alpha \in \Lambda$ and $u_\alpha \uparrow u$.

We say a net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ is *order convergent* to a point a of L if there exist two nets $\{u_\alpha\}_{\alpha\in\Lambda}$ and $\{v_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$ such that

$$a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a.$$

Let $\mathcal{F} = \{F : F = \emptyset$ or $F \subseteq L$ and for each net $\{a_\alpha\}_{\alpha\in\Lambda}$ of F such that if $\{a_\alpha\}_{\alpha\in\Lambda}$ is order convergent to a, then $a \in F\}$.

It is easy to prove that $\emptyset, L \in \mathcal{F}$ and if $F_1, F_2, \ldots, F_n \in \mathcal{F}$, then $\bigcup_{i=1}^n F_i \in \mathcal{F}$, if $\{F_\mu\}_{\mu\in\Omega} \subseteq \mathcal{F}$, then $\bigcap_{\mu\in\Omega} F_\mu \in \mathcal{F}$. Thus, the family $\mathcal{F}$ of subsets of L define a topology τ_0^L on $(L, \perp, \oplus, 0, 1)$ such that $\mathcal{F}$ consists of all closed sets of this topology. The topology τ_0^L is called the *order topology* of $(L, \perp, \oplus, 0, 1)$ (Birkhoff, 1948).

We can prove that the order topology τ_0^L of $(L, \perp, \oplus, 0, 1)$ is the finest (strongest) topology on L such that for each net $\{a_\alpha\}_{\alpha\in\Lambda}$ of $(L, \perp, \oplus, 0, 1)$, if $\{a_\alpha\}_{\alpha\in\Lambda}$ is order convergent to a, then $\{a_\alpha\}_{\alpha\in\Lambda}$ must be topology τ_0^L convergent to a. But the converse is not true.

Lemma 1. *(Junde et al., 2003). Let* $(L, \perp, \oplus, 0, 1)$ *be a totally ordered effect algebra. If* $A = \{a_k\}_{k\in\mathbf{N}}$ *is orthogonal* $\oplus$*-summable, then* $\{a_n\}_{n\in\mathbf{N}}$ *is order topology* τ_0^L *convergent to 0.*

Lemma 2. *(Junde et al., 2003). If* $(L, \perp, \oplus, 0, 1)$ *is a* σ*-complete totally order connect effect algebra, then for each* $h \in L$, $0 < h$, *there exists an orthogonal* $\oplus$*-summable sequence* $\{h_i\}$ *of* L *such that* $\vee_{n\in\mathbf{N}}\{\oplus_{i=1}^n h_i\} < h$.

3. MAIN THEOREM AND ITS PROOF

Definition 3.1. (Junde *et al.*, 2003). Let $(L, \perp, \oplus, 0, 1)$ be a totally ordered effect algebra. We say that the sequence $\{a_n\}_{n\in\mathbf{N}}$ of $(L, \perp, \oplus, 0, 1)$ is an order topology τ_0^L-*Cauchy sequence*, if for each $h \in L$, $0 < h$, there exists $n_0 \in \mathbf{N}$ such that when $n_0 \leq n$, $n_0 \leq m$, if $a_n \leq a_m$, then $a_m \ominus a_n < h$, if $a_m \leq a_n$, then $a_n \ominus a_m < h$.

Let $(L_1, \perp, \oplus, 0, 1)$, $(L_2, \perp, \oplus, 0, 1)$ be two effect algebras. A mapping $\mu : L_1 \to L_2$ is said to be a *measure* which is defined on $(L_1, \perp, \oplus, 0, 1)$ and take valued in $(L_2, \perp, \oplus, 0, 1)$, if $a, b \in (L_1, \perp, \oplus, 0, 1)$ with $a \perp b$, then $\mu(a) \perp \mu(b)$ and $\mu(a \oplus b) = \mu(a) \oplus \mu(b)$. We say the measure μ is *s-bounded* with respect to the order topology $\tau_0^{L_2}$ if for each orthogonal sequence $\{a_n\}$ of $(L_1, \perp, \oplus, 0, 1)$, $\{\mu(a_n)\}$ is order topology $\tau_0^{L_2}$ convergent to 0. If $\{\mu_n : n \in \mathbf{N}\}$ is a sequence of s-bounded L_2-valued measures which are defined on L_1 and for each orthogonal sequence $\{a_k\}$ of $(L_1, \perp, \oplus, 0, 1)$, $\{\mu_n(a_k)\}$ is order topology $\tau_0^{L_2}$ convergent to 0 uniformly with respect to $n \in \mathbf{N}$, then $\{\mu_n : n \in \mathbf{N}\}$ is said to be *uniformaly s-bounded*.

Brooks and Jewett (1970) proved the following famous Vitali–Hahn–Saks measure theorem:

Theorem 1. *If $\mathcal{A}$ is a σ-algebra on a nonempty set Ω, $(X, \|\cdot\|)$ is a Banach space, $\{\mu_n : n \in \mathbf{N}\}$ is a sequence of s-bounded X-valued measures which are defined on $\mathcal{A}$ and for each $A \in \mathcal{A}$, $\{\mu_n(A)\}_{n\in\mathbf{N}}$ is a $\|\cdot\|$ convergent sequence,* $\lim_{\mu_n(A)=0\mu(A)\to 0}$ *for each $n \in \mathbf{N}$, where ν is a nonnegative finite additive measure which is defined on $\mathcal{A}$, then* $\lim_{\mu_n(A)=0\mu(A)\to 0}$ *uniformly with respect to $n \in \mathbf{N}$.*

An interesting problem is whether Theorem 1 is also hold for quantum logics valued measures? Now, we show that the answer is true.

At first, by Lemma 1, 2, and the methods of Junde *et al.* (2003), Junde and Zhihao (2003), and Mazario (2001), we may prove the following lemma:

Lemma 3. *Let $(L_1, \perp, \oplus, 0, 1)$ and $(L_2, \perp, \oplus, 0, 1)$ be two quantum logics, L_1 have the sequentially completeness property, $(L_2, \perp, \oplus, 0, 1)$ be a σ-complete totally order connect effect algebra, $\{\mu_n : n \in \mathbf{N}\}$ be a sequence of s-bounded $(L_2, \perp, \oplus, 0, 1)$-valued measures which are defined on $(L_1, \perp, \oplus, 0, 1)$. If for each $a \in (L_1, \perp, \oplus, 0, 1)$, $\{\mu_n(a)\}_{n\in\mathbf{N}}$, is an order topology $\tau_0^{L_2}$ Cauchy sequence, then $\{\mu_n\}$ is uniformly s-bounded.*

Our main result is:

Theorem 3. *Let $(L_1, 0, 1)$ be a Boolean algebra and $(L_2, \perp, \oplus, 0, 1)$ be a quantum logics, L_1 have the sequentially completeness property, $(L_2, \perp, \oplus, 0, 1)$ be a*

σ-complete totally order connect effect algebra, $\{\mu_n : n \in \mathbf{N}\}$ be a sequence of s-bounded $(L_2, \perp, \oplus, 0, 1)$-valued measures which are defined on $(L_1, 0, 1)$. If for each $a \in (L_1, 0, 1)$, $\{\mu_n(a)\}_{n\in\mathbf{N}}$ is an order topology $\tau_0^{L_2}$ Cauchy sequence, and for each $n \in \mathbf{N}$, when $\{\nu(a)\}$ convergent to 0, $\{\mu_n(a)\}$ is order topology $\tau_0^{L_2}$ convergent to 0, where ν is a nonnegative finite additive measure which is defined on $(L_1, 0, 1)$, then when $\{\nu(a)\}$ convergent to 0, $\{\mu_n(a)\}$ are order topology $\tau_0^{L_2}$ convergent to 0 uniformly with respect to $n \in \mathbf{N}$.

Proof: If the conclusion is not ture, there exists $h \in L_2$, orthogonal $\oplus$-summable sequence $\{h_k\} \subseteq L_2$, $\{n_k\} \subseteq \mathbf{N}$, positive numbers sequence $\{\delta_k\}$, $\{a_k\} \subseteq L_2$, such that $\oplus_{i=1}^{\infty} h_i < h$, for each $k \in \mathbf{N}$, $\oplus_{i=k+1}^{\infty} h_i < h_k$, $\mu_{n_{k+1}}(a_{k+1}) > h$; $\nu(a_{k+1}) < \delta_{k+1}$ and $\nu(a) < \delta_{k+1}$ implies that $\mu_{n_j}(a) < h_{2^{k+3}}$ for each $j \leq k$. Without loss generality, we may assume that $n_i = i$. Thus we have

$$\mu_{k+1}(a_{k+1}) > h, \tag{1}$$

$$\mu_j(a) < h_{2^{k+3}},\ j \leq k, a \leq a_{k+1}. \tag{2}$$

Let $c_1 = a_2$ and $i_1 = 2$. If there exists an $i_2 > 2$ such that $\mu_{i_2}(c_1 \wedge a_{i_2}) > h_4$, then let $c_2 = c_1 \wedge a'_{i_2}$. If $c_1, \cdots, c_k$ and $i_i, \cdots, i_k$ have been chosen and that there exists an $i_{k+1} > i_k$ such that $\mu_{i_{k+1}}(c_k \wedge a_{i_{k+1}}) > h_4$, then let $c_{k+1} = c_k \wedge a'_{i_{k+1}}$. Thus, we have

$$c_{k+1} \leq c_k,\ c_k \ominus c_{k+1} = c_k \wedge a_{i_{k+1}}. \tag{3}$$

It follows from (2), (3), and the assumption of $\{h_i\}$ that

$$\mu_{i_{k+1}}(c_k \wedge c_{k+1}) > h_4. \tag{4}$$

$$\mu_{i_k}(c_k \wedge c_{k+1}) > h_{2^{k+3}}. \tag{5}$$

Now, we show that there exists a $c_{k_0} \in L_1$ and an i_{k_0} such that for all $j > i_{k_0}$, $\mu_j(c_{k_0} \wedge a_j) < h_4$.

In fact, if not, we can obtain an orthogonal sequence $\{c_k \wedge c_{k+1}\}$ of L_1 which satisfy (4) and (5). Thus, we have

$$\mu_{i_{k+1}}(c_k \wedge c_{k+1}) \ominus \mu_{i_k}(c_k \wedge c_{k+1}) > h_8, k = 1, 2, \cdots .$$

This contradicts Lemma 3. Hence, there exists a $c_{k_0} \in L_1$ and an i_{k_0} such that for all $j > i_{k_0}$, $\mu_j(c_{k_0} \wedge a_j) < h_4$.

Let $p_1 = i_{k_0}$, $g_1 = c_{k_0}$, $\mu_i^{(1)} = \mu_{p_1+i}$, $a_i^{(1)} = a_{p_1+i} \wedge g'_1$. It follows from (1) and (5) easily that

$$\mu_1(g_1) < h_{16},$$

$$\mu_2(g_1) < h \ominus h_4.$$

So

$$\mu_2(g_1) \ominus \mu_1(g_1) > h \ominus h_4 \ominus h_{16},$$
$$\mu_i^{(1)}\left(a_i^{(1)}\right) > h \ominus h_4,$$
$$\mu_j^{(1)}(a) < h_{2^{i+3}}, a \leq a_i^{(1)}, j < i.$$

Let $c_1^{(1)} = a_2^{(1)}$. Similarly, we can obtain a $c_{k_1}^{(1)}$ and an i_{k_1} such that for all $j > i_{k_1}$, $\mu_j^{(1)}(c_{k_1}^{(1)} \wedge a_j^{(1)}) < h_8$.

Let $p_2 = i_{k_1}$, $g_2 = c_{k_1}^{(1)}$, $\mu_i^{(2)} = \mu_{p_2+i}^{(1)}$, $a_i^{(2)} = a_{p_2+i}^{(1)} \wedge g_2'$. Then $g_1 \wedge g_2 = 0$ and $\mu_2(g_2) < h_{32}$, $\mu_1^{(1)}(g_2) > h \ominus h_4 \ominus h_8$. So,

$$\mu_1^{(1)}(g_2) \ominus \mu_2(g_2) > h \ominus h_4 \ominus h_8 \ominus h_{32},$$
$$\mu_i^{(2)}\left(a_i^{(2)}\right) > h \ominus h_4 \ominus h_8,$$
$$\mu_j^{(2)}(a) < h_{2^{i+3}}, a \leq a_i^{(2)}, j < i.$$

Inductively, we can obtain an disjoint sequence $\{g_k\}$ of L_1 and $\{\mu_1^{(k)}\}$ such that $\mu_1^{(k+1)}(g_k + 2) \ominus \mu_1^{(k)}(g_k + 2) > h \ominus h_4 \ominus h_8 \ominus \cdots \ominus h_{2^{k+1}} \ominus \cdots > h \ominus h_1$ for all $k \in \mathbf{N}$.

This contradicts Lemma 3 and so the theorem is proved. □

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